Frequency Scaling at the Onset of Finite Amplitude Oscillation *

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The onset of oscillation at non-zero finite amplitude is shown to be governed in the generic case by a scaling law for the frequency of oscillation f in terms of the bifurcation parameter $\mu - \mu_c$, namely $f \sim (\mu - \mu)_c^{1/2}$. This law can be expected to hold whenever such a bifurcation occurs in a nondegenerate system without hysteresis.

A dynamical system which is initially at stable equilibrium can evolve to stable periodic oscillation in at least two distinct generic ways. The amplitude of oscillation may increase continuously from zero; this situation is described by the Hopf bifurcation [1], transpiring locally in phase space. Alternatively, the frequency of oscillation may increase continuously from zero. In the latter case, because the amplitude of oscillation is typically finite at the onset threshold, the phase space structure of this second bifurcation is inevitably global. Furthermore, this bifurcation is discontinuous in the sense of Zeeman [2] (cf. [3, 4]) because the attractor in phase space jumps from dimension 0 (a point) to dimension 1 (a cycle). If such a bifurcation occurs without hysteresis, then at the bifurcation threshold the formerly stable state of equilibrium lies topologically within the newly stable periodic orbit. In the generic case, this involves a saddle-node bifurcation on a cycle. Apart from this case or the Hopf bifurcation, generic bifurcation from equilibrium to limit cycle typically involves hysteresis, as in the case of a saddle connection [5].

The topological phase portrait of the saddle-node on a limit cycle is well-known in the qualitative geometric theory of second-order differential equations [5]. A generic form of this bifurcation is exhibited by the system of two first-order equations [2]

$$\dot{\theta} = \mu - r \cos \theta, \quad \dot{r} = r - r^3; \tag{1}$$

which bifurcates at $\mu_c = 1$ with the phase portraits sketched in Figure 1. For $\mu < \mu_c$, there is an attracting equilibrium at $r = 1$, $\theta = -\cos^{-1} \mu$. Since $\cos^{-1}(1-h) = (2h)^{1/2} + O(h^{3/2})$, we note that to lowest order the distance in phase space separating the saddle from the node varies as $(\mu_c - \mu)^{1/2}$, governed by the normal form of the saddle-node [6].

For $\mu > \mu_c$ in (1), there is a stable limit cycle with $r \equiv 1$. The frequency of oscillation decreases to zero as μ approaches μ_c from above. In fact, the period T of oscillation can be found by integrating (1); upon separating variables, one obtains

$$T = \int_{-\pi}^{\pi} d\theta / (\mu - \cos \theta).$$

A short computation shows that to lowest order in $\mu - \mu_c$ this is

$$T = 2\pi \sqrt{2} \, (\mu - \mu_c)^{-1/2},$$

so the frequency $f = 1/T$ varies as

$$f \sim (\mu - \mu_c)^{1/2}.$$

The same scaling has recently been observed in onset of a finite amplitude oscillation in fluid convection [7].

It may be noted that this scaling of frequency is the same as the scaling of amplitude at a Hopf bifurcation. The same scaling also occurs in a type of transition to chaotic attractor by temporal intermittency, that is, the Type 1 intermittency of Pomeau and Manneville [8].

There is in fact a topological analogy between such intermittency and the phase portraits of Figure 1. To see this, we may first imagine this figure to represent a phase portrait of a plane Poincaré map rather than a planar flow; we obtain a bifurcation by

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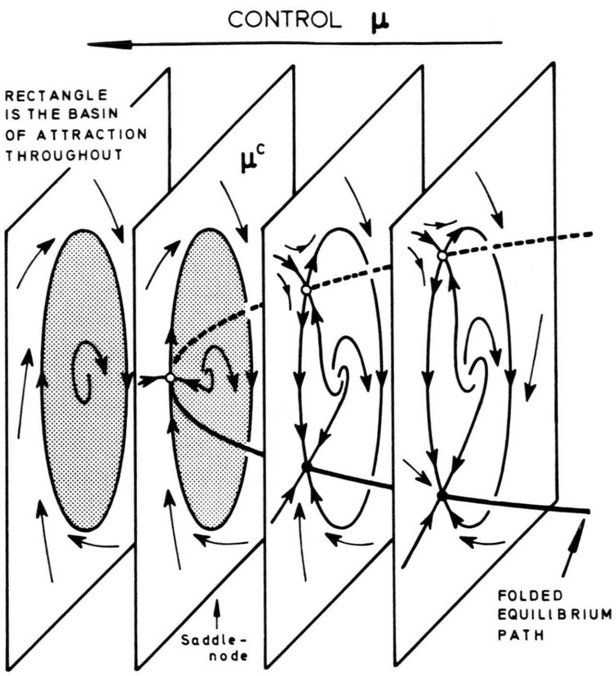

Fig. 1. Successive phase portraits under varying μ at the onset of finite amplitude oscillation from equilibrium via saddle-node bifurcation.

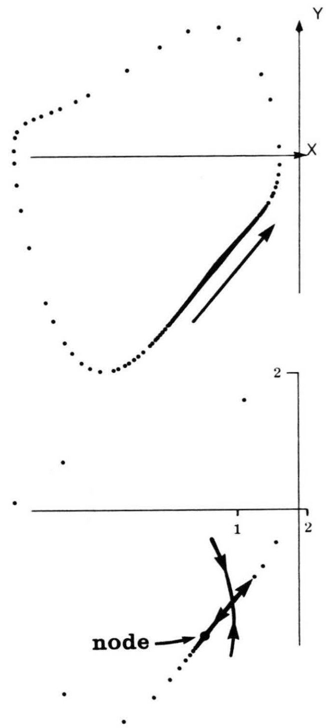

Fig. 2. Poincaré sections of the forced van der Pol oscillator eqs. (2) before and after mode locking via cyclic saddle-node.

cyclic saddle-node on a torus, or in other words, the onset of a second finite amplitude oscillation with incommensurate frequency. (Varying the bifurcation parameter in the opposite sense, we have mode locking.) The difference between this and Type 1 Pomeau-Manneville intermittency is that the torus is replaced by a homoclinic tangle [9, 10].

As an example of the cyclic saddle-node on a torus, consider the periodically forced van der Pol system

$$\dot{y} = (1 - x^2)\, y - x - A \sin(1.1\, t), \quad \dot{x} = y \qquad (2)$$

which locks to the forcing frequency by cyclic saddle-node bifurcation near $A \simeq 0.6125$. The phase portraits in the Poincaré section at $t = 0$ are shown in Figure 2. The upper part shows quasiperiodic motion at $A = 0.61$ represented by return points on a section of a torus; below at $A = 0.62$ is a portrait with node attractor and saddle nearby.

The scaling of frequency in this example was determined by numerical integration of (2), and the results are shown in Fig. 3, where the ratio of

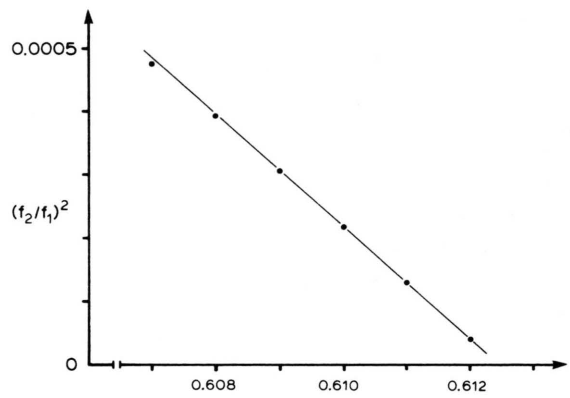

Fig. 3. Dependence of frequency ratio f_2/f_1 squared upon bifurcation parameter A near mode locking in eqs. (2).

drift frequency to forcing frequency is plotted against forcing amplitude A, and the square-root scaling law is confirmed.

Finally, we note that an oscillation may also slow down to zero frequency, with the dynamical system subsequently making a sudden, rapid jump to a remote attractor, or diverging to infinity. (In the case of jump to a remote attractor, there will typically be hysteresis.) Such slowing down has a topologically different phase portrait from the saddle-node on a cycle, being instead described by a limit cycle approaching a nearby saddle point to form a structurally unstable homoclinic connection [5, 6, 10]. In this case, the frequency will not in general scale by the square root law, but will be governed by the local characteristic exponents of the homoclinic saddle.

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